Corrections (Or explanations)

1) Recall (Formal statement, Chebyshev's Inequality): Let X be a real-val r.v, let $\varphi : \mathbb{R} \to \mathbb{R}$ be nonnegative and nondecreasing, let $a \in \mathbb{R}$ and suppose that $\varphi(a) > 0$, Then

$$P(X \geq a) \leq \frac{E\varphi(X)}{\varphi(a)}$$

Special case

Let $\varphi(x) = \begin{cases} x^2, & x \geq 0 \\ 0, & x \leq 0 \end{cases}$ Then φ is nonnegative, nondecreasing

Now let $a > 0$ Then

$$P(X \geq a) \leq \frac{E\varphi(X)}{\varphi(a)} \leq \frac{EX^2}{a^2}$$

2) Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}_{[0,1]}, m)$ Let $\lambda > 0$. Define

$X : \Omega \to \mathbb{R}$ by $X(\omega) = -\frac{1}{\lambda} \ln(\omega)$

X is measurable. For $r \geq 0$, we have

$$\{X > r\} = \{\omega \in [0,1] : X(\omega) > r\} = \{\omega \in [0,1] : \ln(\omega) < -\lambda r\}$$
$$= \{\omega \in [0,1] : \omega < e^{-\lambda r}\} = [0, e^{-\lambda r})$$ And

For $r \leq 0$ we have

$$\{X \leq r\} = \{\omega \in [0,1] : X(\omega) \leq r\} = \{\omega \in [0,1] : -\frac{1}{\lambda} \ln(\omega) \leq r\}$$
$$= \{\omega \in [0,1] : \ln(\omega) \geq -r\lambda\} = \{\omega \in [0,1] : \omega \geq e^{-\lambda r}\} \cancel{([e^{-\lambda r}, \infty))}$$
$$= \emptyset \in \mathcal{B}_{[0,1]}$$

So X is measurable for $r \in \mathbb{R}$.

when $r \geq 0$, $P(X \leq r) = 1 - P(X > r) = 1 - P([0, e^{-\lambda r})) = 1 - e^{-\lambda r}$

when $r \leq 0$, $P(X \leq r) = P(\emptyset) = 0$

Ceren Vardar Acar (1)

Explanation 2

Recall: Thm: Let $X, X_1, X_2, \dots$ be real-valued random variables with distribution functions $F, F_1, F_2, \dots$ then

i) $F_n(x) \to F(x)$ for all x at which F is continuous

iff ii) $Eg(X_n) \to Eg(X)$ for all bounded, continuous g.

iff iii) $\int_{\mathbb{R}} \mu_n(dx) g(x) \to \int_{\mathbb{R}} \mu(dx) g(x)$ for all bounded continuous g

iff iv) .. There exists Y, Y_n having the same distribution as X, X_n and for which $Y_n \to Y$ almost surely.

Question How do we connect iv) with the rest of the items?

1) iv) implies ii)

Lemma Let $X, X_1, X_2, \dots$ be real-valued r.v.s if $X_n \to X$ a.s then $X_n \to X$ in probability by the Dominated convergence theorem. Let $\varepsilon > 0$

$P(|X_n - X| > \varepsilon) = E 1\{|X_n - X| > \varepsilon\}$ (these rv's are bdd by 1 and $E1 < \infty$)

as $n \to \infty$ these rv's converge to 0 a.s. By DCT

$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = E\,0 = 0$ as required

Proposition Suppose $X_n \to X$ in probability and $f: \mathbb{R} \to \mathbb{R}$ is cont, bdd

Then $Ef(X_n) \to Ef(X)$ as $n \to \infty$.

Let n_k be a subsequence. Then X_{n_k} converges to X in probability

So there exists a subsequence $X_{n_{k_l}}$ converging to X a.s by (Thm 1.6.2 Durrett)

By continuity, $f(X_{n_{k_l}})$ converges to $f(X)$ a.s. And by Bounded Convergence Thm, $Ef(X_{n_{k_l}}) \to Ef(X)$ as $l \to \infty$.

Thus for every subsequence $Ef(X_{n_k})$ of $Ef(X_n)$ there exists a further " $Ef(X_{n_{k_l}})$ converges to $Ef(X)$

Thus by (Lemma 1.6.3 Durrett) $Ef(X_n) \to Ef(X)$ as $n \to \infty$

In the explanations given above replace X_n & X by Y_n & Y and convergence in distribution is the satisfied for Y_n & Y and as they have same distribution as X_n & X the result holds.

2) Now i) $\to$ iv) by (Thm 2.2.1 Durrett) there exist Y_n with the same distr as X_n that conv a.s to Y which has the same distr as X.

Proof: (Thm 2.2.1, Durrett)

Let U be a Uniform $(0,1)$ r.v defined on some prob space $(\Omega, \mathcal{F}, P)$

Let $F_n^{-1}(u) = \sup(y : F_n(y) < u) = a_n(u)$ and same for F^{-1}.

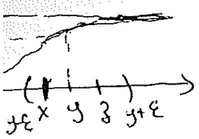
Set $Y_n = F_n^{-1}(u)$ and $Y = F^{-1}(u)$. Then $F_n^{-1}(u) \to F^{-1}(u)$ for all u for which $a(u) = b(u)$ where $b(u) = \inf(x : F(x) > u)$

Let u be such a point. Let $\varepsilon > 0$ and set $y = F^{-1}(u)$

Let $x \in (y-\varepsilon, y)$ and $z \in (y, y+\varepsilon)$ be places where F is continuous (F jumps at countably many places)

Then $F(x) < u < F(z)$

And $F_n(x) \to F(x)$, $F_n(z) \to F(z)$ then there exits N s.t if $n > N$, $F_n(x) < u < F_n(z)$ Thus if $n > N$,



$x < F_n^{-1}(u) \leq z$ by def of $F_n^{-1}(u)$

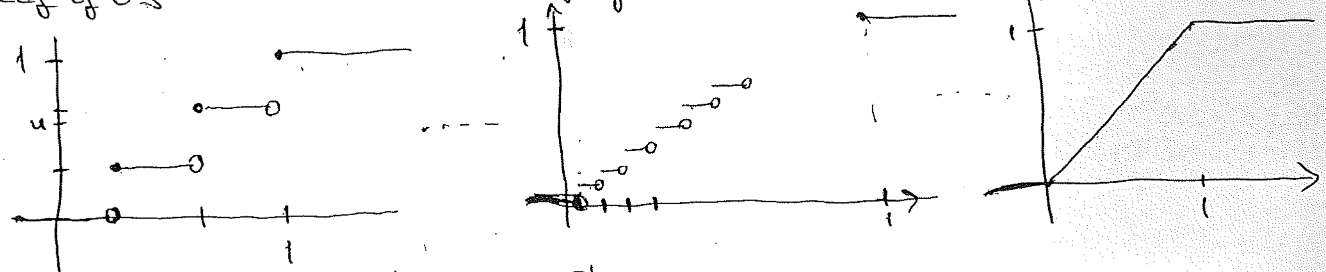
$x \leq a_n(u) \leq b_n(u) \leq z$

That is for all $n > N$, $y - \varepsilon < F_n^{-1}(u) < y + \varepsilon$, which means $F_n^{-1}(u) \to F^{-1}(u) = y$ as $n \to \infty$.

With prob 1 $a(u) = b(u)$ since it fail for only countably many values of u. Thus with prob one, $F_n^{-1}(u) \to F^{-1}(u)$ hence $Y_n \to Y$ a.s.

Now the relation with the example is:

cdf of U_3 — cdf of U_{10} — cdf of u



For all u, $Y_n = F_n^{-1}(u) \to u = F^{-1}(u) = Y$ so we have a.s convergence.